EFFECT OF ANISOTROPY OF THE MEDIUM ON THE STRESSES ON THE SURFACE OF A CRACK

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In this article, we solve a three-dimensional problem concerning the effect of anisotropy on the stress concentration on the surface of a narrow crack in orthotropic and transversely isotropic elastic media located in a uniform external field. By a narrow crack, we mean an elliposoidal cavity characterized by two small (but finite) parameters. Explicit expressions are obtained for the stresses on the surface of the crack in relation to the elastic constants of the medium and is a normal to the surface for different external fields. It is established that anisotropy of the medium produces quantitative but not qualitative changes in the stress distribution on the crack. For example, anisotropy allows the stress maximum to be shifted away from the edge of the crack when the material is subjected to uniaxial tension. We determine the conditions that must be satisfied by the elastic constants of the medium for such a shift to occur. We also calculate the coordinates of the points of different maxima and the values of the maximum stresses. We establish the conditions under which the stresses in the ends of the crack will be greater than the stresses at the vertices of the middle section. Such results are basically impossible to obtain by solving the corresponding two-dimensional problem.

In our study, we make use of the general solution given in [1] for the problem of the stress distribution on the surface of an ellipsoidal cavity.

1. We will examine an ellipsoidal crack in an elastic medium under the influence of a uniform external field $\sigma_0{}^{\alpha\beta}$. We rigidly connect the system of coordinates $x^\alpha(\alpha = 1, 2, 3)$ with the semi-axes a_1, a_2, a_3 of the ellipsoid and we introduce the dimensionless parameters $\zeta = a_2 a_1{}^{-1}$, $\xi = a_3 a_2{}^{-1}$. The case $\zeta \ll 1$, $\xi \sim 1$ corresponds to a needle, the case $\zeta \sim 1$, $\xi \ll 1$ corresponds to a crack, and the case $\zeta \ll 1$, $\xi \ll 1$ corresponds to a narrow crack. We will obtain the solution for a narrow crack from the solution for a needle [2], having expanded it in the small parameter ξ and having limited ourselves to the principal terms of the expansion.

The stresses on the surface of an arbitrary ellipsoidal cavity are found from the formulas [1]

$$\sigma^{\alpha\beta}(\mathbf{n}) = B^{\alpha\beta\lambda\mu}(\mathbf{n}) B^{-1}_{\lambda\mu\varkappa\rho}\sigma_0^{\varkappa\rho},$$
$$B^{\alpha\beta\lambda\mu}(\mathbf{n}) = c^{\alpha\beta\lambda\mu} - c^{\alpha\beta\varkappa\rho}K_{\varkappa\rho\nu\eta}(\mathbf{n})\,c^{\nu\eta\lambda\mu}.$$

Here, $\mathbf{n} = (n_1, n_2, n_3)$ is a unit normal to the surface of the ellipsoid; $c^{\alpha\beta\lambda\mu}$ is the tensor of the elastic constants of the medium; $B^{-1}_{\lambda\mu\kappa\rho}$ is a constant tensor which is the inverse of $B^{\lambda\mu\kappa\rho}$. For a needle, the latter has the form [2]

$$B = B_0 + O(\xi^2 \ln \zeta), \quad B_0 = \frac{\xi}{2\pi}\int_0^\pi \frac{B(\varphi, 0)\,d\varphi}{\cos^2\varphi + \xi^2 \sin^2\varphi}, \tag{1.1}$$
$$B(\varphi, 0) = B(\mathbf{n})|_{n_1=0,\ n_2=\cos\varphi,\ n_3=\sin\varphi}.$$

The components of the tensor B(n) in the principal sections of an ellipsoid were presented in [2] for an orthotropic medium, while the components of K(n) over the entire surface of the ellipsoid were given in [3]. In order to pass to the limit for a narrow crack, in (1.1) we examine the function

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TABLE 1

Stress	Axis of elastic symmetry	Section	Conditions for the elastic parameters of the medium for which the stress maxima are displaced
σ_1	x^1 x^2	$n_2 = 0$ $n_1 = 0$	$\varkappa < \frac{\nu_2(1+\nu_1)}{2\nu_1}$
	x^3	$n_1 = 0$ $n_2 = 0$	$\varkappa < \nu_2 + \frac{\nu_1(1-\nu_1)k_0^2}{2\nu_2}$
	x^1 x^2	$n_3 = 0$	$\varkappa < \frac{\nu_1(1-\nu_1)+\nu_2(1+\nu_1)}{2\nu_1}, \quad k_0^2 \geqslant \frac{\nu_2}{\nu_1}$ $\varkappa < \nu_2 + \frac{(1-\nu_1)(\nu_1+\nu_2)k_0^2}{2\nu_2}, \quad k_0^2 < \frac{\nu_2}{\nu_1}$
σ_2	x^1 x^2	$n_2 = 0$ $n_1 = 0$	$\varkappa < \nu_2 + \frac{1-\nu_1}{\nu_2}$
	x^3	$n_1 = 0$ $n_2 = 0$	$\varkappa < \nu_2 + \frac{(1-\nu_1)k_0^2}{2}$
	x^1 x^2	$n_3 = 0$	$\varkappa < \frac{\rho\nu_1^2+\nu_2}{\nu_2(1+\nu_1)}, \quad \rho < \frac{\nu_2^2}{\nu_1^2}$ $\varkappa < \frac{\nu_2}{\nu_1}, \quad \rho \geqslant \frac{\nu_2^2}{\nu_1^2}$

$$f(\varphi, \xi) = \pi^{-1}\xi(\cos^2\varphi + \xi^2\sin^2\varphi)^{-1}.$$

It can be shown that at $\xi \to 0$, $\phi \to \pi/2$ $f(\phi, \xi) \to \delta(\phi - \pi/2)$, where $\delta(\phi)$ is the Dirac delta function. Thus, we write the expansion of $f(\phi, \xi)$ in the small parameter ξ as

$$f(\varphi, \xi) = \delta(\varphi - \pi/2) + \xi \cos^2\varphi + O(\xi^2).$$

We obtain the expansion of the tensor B in ξ from (1.1) after we insert this equation into the expansion of $f(\phi, \xi)$ and regularize the integrals [4]:

$$B = B_{00} + \xi B_{01} + O(\xi^2, \xi^2 \ln \xi), \quad B_{00} = B(\pi/2, 0),$$
$$B_{01} = \frac{2}{\pi}\int_0^{\pi/2} [B(\varphi, 0) - B(\pi/2, 0)] \cos^{-2}\varphi d\varphi.$$

Thus, determination of the stresses on the surface of the crack reduces to finding single integrals and inverting the tensor B.

2. We will assume that the external medium is orthotropic and that the axes of elastic symmetry of the medium coincide with the axes of the ellipsoid. Omitting the details of the calculation and the inversion of B, we present the components of the tensor B^{-1}. In the given case, we retain two principal terms of the expansion in the small parameter ξ:

$$\begin{gathered}
B_{11\alpha\alpha}^{-1} = \Delta_{1\alpha}\Delta^{-1} \quad (\alpha = 1, 2, 3), \quad B_{2222} = \Delta_{22}\Delta^{-1}, \\
B_{2233}^{-1} = \Delta_{11}^{-1}(\Delta_{12}\Delta_{13}\Delta^{-1} - kc_{33}), \\
B_{3333}^{-1} = \xi^{-1}L\Delta_{11}^{-1} + \Delta_{33}\Delta^{-1}, \quad B_{1212} = (4c_{66})^{-1}, \\
B_{1313}^{-1} = (4c_{55})^{-1}\left(\xi^{-1}\sqrt{c_{55}c_{66}^{-1}} + 1\right), \quad B_{2323}^{-1} = (4k\Delta_{11})^{-1}L(\xi^{-1} + l), \\
k = \sqrt{\frac{c_{22}}{c_{33}}}, \quad l = \sqrt{\frac{\Delta_{11} - 2c_{23}c_{44}}{c_{33}c_{44}} + 2k}, \quad L = klc_{33}.
\end{gathered} \tag{2.1}$$

Here $c_{\alpha\beta} = c^{\alpha\alpha\beta\beta}$ (α, β = 1, 2, 3); $c_{44} = c^{2323}$; $c_{55} = c^{3131}$; $c_{66} = c^{1212}$; Δ, $\Delta_{\alpha\beta}$ are the determinant and the algebraic complements of the elements $c_{\alpha\beta}$ of the matrix $\|c_{\alpha\beta}\|$ (α, β = 1, 2, 3); k and ℓ are anisotropic parameters analogous to those introduced in [5] for plane problems.

The behavior of the stresses will be studied in the local coordinate system $x^{\alpha'}$ (α' = 1, 2, 3) with the axis $x^{3'}$ directed along a normal to the surface [6]. By virtue of the equilibrium conditions, in the new axes the only nontrivial stresses will be be $\sigma^{1'1'} = \sigma_1$, $\sigma^{2'2'} = \sigma_2$ and $\sigma^{1'2'} = \tau$. Meanwhile, σ_1 and σ_2 in sections $n_1 = 0$ and $n_2 = 0$ are directed perpendicular to the plane and along the contour of the section on the edge of the crack. They have the opposite direction in the section $n_3 = 0$.

Due to the cumbersome nature of the general expressions for σ_1, σ_2, and τ, we will examine only the singular components of these stresses along the principal sections of the crack. In the section $n_1 = 0$ we have

$$\sigma_1 = -(\xi\Delta_{11})^{-1}L\eta_1\left(n_2^2\sigma_0^{33} - k^{-1}n_2n_3\sigma_0^{23}\right),$$
$$\sigma_2 = -\Delta_{11}p_1\eta_1^{-1}\sigma_1, \quad \tau = \left(\xi\sqrt{c_{55}c_{66}}\right)^{-1}\psi_1 n_2\sigma_0^{13}; \tag{2.2}$$

in the section $n_2 = 0$

$$\sigma_1 = -\xi^{-1}\eta_2\left[L\Delta_{11}^{-1}n_1^2\sigma_0^{33} - \left(\sqrt{c_{55}c_{66}}\right)^{-1}n_1n_3\sigma_0^{13}\right],$$
$$\sigma_2 = -\Delta_{22}p_2\eta_2^{-1}\sigma_1, \quad \tau = (\xi k\Delta_{11})^{-1}L\psi_2\sigma_0^{23}; \tag{2.3}$$

and in the section $n_3 = 0$

$$\sigma_1 = -(\xi\Delta_{11})^{-1}L\eta_3\sigma_0^{33}, \quad \sigma_2 = p\eta_3^{-1}\sigma_1,$$
$$\tau = \xi^{-1}\psi_3\left[(k\Delta_{11})^{-1}Ln_1\sigma_0^{23} - \left(\sqrt{c_{55}c_{66}}\right)^{-1}n_2\sigma_0^{13}\right]. \tag{2.4}$$

Here

$$p = p_3\left[\Delta_{22}n_1^4 + \Delta_{11}n_2^4 + \left(\Delta c_{66}^{-1} + 2\Delta_{12}\right)n_1^2n_2^2\right],$$
$$p_1 = \left[c_{22}n_2^2 + c_{33}n_3^2 + \left(\Delta_{11}c_{44}^{-1} - 2c_{23}\right)n_2^2n_3^2\right]^{-1},$$
$$\eta_1 = p_1\left(\Delta_{13}n_2^2 + \Delta_{12}n_3^2\right), \quad \psi_1 = c_{55}c_{66}\left(c_{66}n_2^2 + c_{55}n_3^2\right)^{-1}. \tag{2.5}$$

The expressions for p_2, η_2, ψ_2 and p_3, η_3, ψ_3 are obtained from p_1, η_1, ψ_1 by the substitutions of indices $1 \leftrightarrow 2$, $4 \leftrightarrow 5$ and $1 \leftrightarrow 3$, $4 \leftrightarrow 6$.

We also determine the stresses at the tips of the crack $A(n_1 = 1,\ n_2 = n_3 = 0)$ and $B(n_1 = n_3 = 0,\ n_2 = 1)$ (there are no singular stresses at the tip $C(n_1 = n_2 = 0,\ n_3 = 1)$):

$$\sigma_1(A) = -\Delta_{23}\Delta_{22}^{-1}\sigma_2(A) = -L\left(\xi c_{11}\Delta_{11}\right)^{-1}\Delta_{23}\sigma_0^{33},$$
$$\sigma_1(B) = -\Delta_{13}\Delta_{11}^{-1}\sigma_2(B) = -L\left(\xi c_{22}\Delta_{11}\right)^{-1}\Delta_{13}\sigma_0^{33},$$
$$\tau(A) = (\xi k\Delta_{11})^{-1}Lc_{44}\sigma_0^{23}, \quad \tau(B) = -\left(\xi\sqrt{c_{66}}\right)^{-1}\sqrt{c_{55}}\,\sigma_0^{13}. \tag{2.6}$$

We find from this that the stresses in the end A in an anisotropic medium (in contrast to an isotropic medium) may be greater than at the lateral point B. Thus, $\sigma_1(A) > \sigma_1(B)$ and $\sigma_2(A) > \sigma_2(B)$ if the following conditions are satisfied

$$c_{11}c_{22}(c_{13} + c_{23}) > c_{12}(c_{11}c_{23} + c_{22}c_{13}), \quad c_{11}c_{23}^2 > c_{22}c_{13}^2.$$

It should be noted that the given result is basically impossible to obtain from the solution of the plane problem.*

3. We will study the qualitative effect of anisotropy of the medium on the behavior of the stresses. Analyzing Eqs. (2.2)-(2.5), we find that in an orthotropic medium in uniaxial tension σ_0^{33} the maximum of the stresses σ_1, σ_2 may be displaced from the edge of the crack. The contrasts with the isotropic case [6], in which the maximum is found on the edge. In the section $n_2 = 0$, the stress σ_2 has a maximum value σ_2^* at the point $n_1^* < 1$ if the following relation is satisfied for the elastic constants of the medium

*The solution of the plane problem corresponds to the values of the stresses in the section $n_1 = 0$, where the ends have almost no effect.

$$c_{55} > \Delta_{22}(c_{11} + 2c_{13})^{-1}. \tag{3.1}$$

Here

$$n_1^* = \sqrt[4]{c_{33}c_{55}\left[(c_{11} + c_{33} + 2c_{13})\,c_{55} - \Delta_{22}\right]^{-1}},$$

$$\sigma_2^* = L\xi^{-1}\left[\Delta_{22}c_{55}^{-1} - 2\,(c_{13} + c_{33}) + 2\sqrt{c_{33}\,(c_{11} + c_{33} + 2c_{13}) - \Delta_{22}c_{55}^{-1}}\right]^{-1}. \tag{3.2}$$

The following condition is sufficient for the maximum of σ_1 to be shifted from the edge of the crack

$$c_{55} > \Delta_{22}\Delta_{23}(c_{11}\,\Delta_{12} + 2c_{13}\Delta_{23})^{-1}. \tag{3.3}$$

In the section $n_1 = 0$, similar relations are obtained from (3.1)-(3.3) by the substitution of indices $2 \leftrightarrow 1$ and $5 \leftrightarrow 4$.

In contrast to the case of an isotropic medium, on the edge of a crack in an orthotropic medium the singular stresses depend on the coordinates of the given point. Meanwhile, the maxima of σ_1 and σ_2 may be shifted along the edge from the tips. The maximum of σ_1 is displaced from the end A when

$$c_{66} > \Delta_{23}\Delta_{33}\left[c_{11}(\Delta_{13} + \Delta_{23}) + 2c_{13}\Delta_{23}\right]^{-1}.$$

If the external field is pure shear σ_0^{13}, then the maximum (with respect to modulus) values of σ_1 and σ_2 in the section $n_1 = 0$ are displaced from the edge of the crack at point n_2^* under the condition $c_{66} > 2c_{55}$. Here,

$$n_2^* = \pm\sqrt{c_{55}\,(c_{66} - c_{55})^{-1}}, \quad \sigma_2^* = \sqrt{c_{66}}\,(2\xi\sqrt{c_{66} - c_{55}})^{-1}.$$

As in the isotropic case, in the section $n_2 = 0$ we see the stress "bump effect" [6].

4. We will study the behavior of the stresses on the a narrow crack in a transversely isotropic medium with the elastic constants E_i, ν_i, G ($i = 1, 2$) in relation to the location of the axis of elastic symmetry of the medium. The relation between the elastic constants $c_{\alpha\beta}$ can be obtained from Hooke's law [5]. The anisotropy parameters k and ℓ for a medium with its symmetry axis directed along x^3 take the form

$$k_0 = \sqrt{\frac{\rho - \nu_2^2}{1 - \nu_1^2}}, \quad l_0 = \sqrt{\frac{E_1}{G\left(1 - \nu_1^2\right)} - \frac{2\nu_2}{1 - \nu_1} + 2k_0}, \quad \rho = \frac{E_1}{E_2}.$$

For the axes of symmetry directed along x^2 and x^1, $k = k_0^{-1}$, $\ell = \ell_0 k_0^{-1}$, and $k = 1$, $\ell = 2$.

We find from Eqs. (2.2)-(2.5) that for the case of uniaxial tension σ_0^{33} in the sections of the crack lying in the planes of isotropy of the medium, the behavior of the stresses is the same as in an isotropic medium: the maximum is reached at points of the edge, and the stresses on the edge are constant.

For the sections passing through the axis of elastic symmetry, the conditions under which the maxima of σ_1 and σ_2 are displaced in tension σ_0^{33} are shown in Table 1. It is evident from the table that the qualitative pattern of behavior of the stresses changes with an increase in G (a decrease in $\kappa = E_1[2G(1 + \nu_1)]^{-1}$).

An analysis of the results and the calculations showed that the shift of the maximum stresses away from the crack edge is most pronounced when the axis of elastic symmetry of the medium is directed perpendicular to the plane of the crack. Here, the magnitude of the maximum and its location depend to a significant extent on the shear modulus G: with an increase in G, the difference between the maximum value and its value on the edge increases, while the maximum point is displaced from the edge.

The stresses at the end of the crack become greater than the stresses at the tip of its middle section only when the axis of elastic symmetry of the medium passes through the plane of the crack. For a symmetry axis directed along x^1, the stresses $\sigma_i(A) > \sigma_i(B)$, $i = 1, 2$ if the following inequalities are satisfied

$$E_1\nu_1 > E_2\nu_2[1+\nu_1(\nu_2-\nu_1)], \quad E_1\nu_1^2 > E_2\nu_2^2.$$

The signs of the inequalities are reversed for the symmetry axis x^2.

We can use (2.6) to obtain conditions by which the stress-concentration factor on a crack in an anisotropic medium becomes greater than in an isotropic medium. In particular, for a transversely isotropic medium with the symmetry axis x^2, this is seen at $\ell_0 > 2$ (for example, it is seen for glass-reinforced plastics).

If the external field is pure shear σ_0^{13}, then in the sections passing through the axis of elastic symmetry the maxima of the stresses σ_1 and σ_2 may be displaced from the edge of the crack — as in uniaxial tension. For media with axes of elastic symmetry directed along x^2 and x_3, in the section $n_1 = 0$ the shift occurs when $\kappa < 0.5$ and $\kappa > 2$, respectively. In the section $n_2 = 0$, the "bump effect" is seen for any direction of the axis of elastic symmetry.

Thus, the results obtained here show when it is expedient to use a three-dimensional model and allow for anisotropy of the medium in the solution of specific problems.

LITERATURE CITED

1. I. A. Kunin and É. G. Sosnina, "Concretization of the stresses on an ellipsoidal discontinuity in an elastic anisotropy medium," Prikl. Mat. Mekh., 37, No. 2, (1973).
2. G. N. Mirenkova and É. G. Sosnina, "Hollow ellipsoidal needle in an orthotropic elastic medium," ibid., 54, No. 6 (1990).
3. G. N. Mirenkova and É. G. Sosnina, "Rigid ellipsoidal disk and needle in an anisotropic elastic medium," ibid., 45, No. 1 (1981).
4. I. M. Gel'fand and G. E. Shilov, Generalized Functions and Operations on Them [in Russian], GIFML, Moscow (1959).
5. S. G. Lekhnitskii, Theory of Elasticity of Anisotropic Bodies [in Russian], Nauka, Moscow (1977).
6. I. A. Kunin, G. N. Mirenkova, and É. G. Sosnina, "Ellipsoidal crack and needle in an anisotropic elastic medium," Prikl. Mat. Mekh., 37, No. 3 (1973).